CSCI 7000-019 Fall 2023: Problem Set 7
Counting Under Symmetry
Due: Monday Nov 27, 2023
Suggested Turn-In Date: Friday Nov 17, 2023

Joshua A. Grochow

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1. Use a group action to give an alternative proof that the number of n-cycles (that is, cyclic orderings of the numbers $\{1, \ldots, n\}$) is $(n-1)!$.

2. We can think of the previous exercise as counting the number of ways to color the directed n-cycle graph C_n with n colors, using every color at least once. Derive how many ways there are to color C_n with $n-1$ colors, using every color at least once.

3. Using group theory, determine how many labeled graphs (using labels $\{1, 2, 3, 4\}$) are isomorphic to the 4-vertex graph that consists of a square and one diagonal (equivalently, the complete graph minus a single edge).

4. Using group theory, determine how many labeled graphs are isomorphic to a rooted complete binary tree of height h (hence, on $2^{h+1} - 1$ vertices).

5. (Cauchy–Frobenius–Burnside Lemma) Let G be a finite permutation group acting on a set Ω. Prove that

$$(\#\text{ orbits of } G \text{ on } \Omega) = \frac{1}{|G|} \sum_{g \in G} |\text{fix}(g)|.$$

   Hint: Consider the set $A = \{(\omega, g) \in \Omega \times G | \omega^g = \omega\}$. Count A in two ways: (1) sum over each orbit the size of the stabilizer of elements in that orbit, and (2) sum over the elements of G, the number of points fixed by g.

6. (Pólya Enumeration Theorem, unweighted case) Let G be a finite permutation group acting on a set Ω. Let C be a finite set (of "colors"), and let Γ be the set of functions $\Omega \to C$ (we can think of each such function as assigning a color to each element of Ω). For $g \in G$, let $c(g)$ be the number of cycles of g on Ω (including 1-cycles, i.e., fixed points). Prove that

$$(\#\text{ orbits of } G \text{ on } \Gamma) = \frac{1}{|G|} \sum_{g \in G} |C|^{c(g)}$$

7. Let V_n be a set (of "vertices") of size, and let $E_n = \{\{u,v\} : u,v \in V_n, u \neq v\}$ be the set of unordered pairs of distinct elements of V_n, and let $A_n = \{(u,v) : u,v \in V_n, u \neq v\}$ be the set of ordered pairs of distinct elements of V_n. Realize that an assignment of the colors $\{\text{black}, \text{clear}\}$ to the set E_n is the same thing as an undirected graph on vertex set V_n, and an assignment of those colors to the set A_n is the same thing as a directed graph on vertex set V_n. Using Pólya's Theorem:
   (a) Compute the number of isomorphism types of undirected graphs on 3 vertices. *Hint:* The answer is 4.
   (b) Compute the number of isomorphism types of undirected graphs on 4 vertices. *Hint:* The answer is 11.
   (c) Compute the number of isomorphism types of *directed* graphs on 3 vertices.

8. (a) Consider the group generated by the n-cycle $(1, 2, 3, \ldots, n)$ (this is known as the cyclic group of order n). For each c, how many group elements are there with exactly c cycles?
   (b) Using Pólya's Theorem and inclusion-exclusion, give another derivation of the results of Exercises 1 and 2.
   (c) How many n-vertex necklaces are there with 2 colors?